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IS THE AVERAGE SECONDARY SCHOOL PUPIL ABLE TO ACQUIRE A THOROUGH KNOWLEDGE OF ALL THE MATHEMATICS ORDINARILY GIVEN IN THESE SCHOOLS?

By I. J. SCHWATT.

Before taking up the subject of our discussion, I ought to state that it has always been with the greatest reluctance that I have felt constrained to criticise the teaching of mathematics in the elementary and secondary schools. The reason for my reluctance is the inequity of the situation. We college teachers have an opportunity to test the knowledge of many who have been taught in the elementary and secondary schools, and on the results of this test, our favorable or unfavorable criticism is based. But the inequity exists because we college teachers constitute ourselves a court of last resort, so to speak. True enough, the college student will be judged by the mental capabilities which he possesses, but very few of them will have to show how much actual knowledge of mathematics they have gained while under our instruction. If we college teachers are unsuccessful with the student, we have the defense that he came to us not fully prepared and that the foundation of his mathematical knowledge was not strong enough to put the superstructure on. However, I feel that there is as much need for improvement in the course of study and the method of teaching

of college mathematics as there is of the elementary and secondary mathematics.

In answering the proposed question, the following assumptions are made:

1. That secondary mathematics has little application in the daily work of the average person. We must not forget that the majority of men and women are artisans, mechanics, clerks or followers of the many other occupations which are indispensable to the common good. I have failed to find, in any of the ordinary pursuits of life, the slightest trace of any direct application of the mathematics taught in the high schools and colleges. Followers of the different occupations soon learn to perform what arithmetical operations they need with facility, and very often in a more effective manner than that which they have practiced at school. It ought to be the ambition of graduates of secondary schools to choose an occupation more in accord with their higher intellectual development and training, but even in the practical life of the lawyer, the physician, etc., hardly any mathematics beyond the ordinary arithmetical operations is necessary. The need for mathematics in business calculations is largely obviated by the common use of calculating machines, tables for interest and discount, for converting domestic money into foreign, and vice versa, etc. But all men, whatever their vocation in life, are in need of those powers of mind which are awakened, strengthened and developed by proper training in mathematics.

2. That the purpose of the secondary school, as well as that of the elementary school, is to develop in the young two principal qualities—character and power of mind. Endowed with these qualities, the individual will be much better equipped for any work and will be more successful in the true sense of the word, than, for instance, if he has learned to make a bread plate when he is to become a salesman in a grocery store or a stenographer in a bank.

It is desirable that the school should train the eye and ear and develop the various latent faculties in the young. The school should train the pupil to be handy with tools, a desirable accomplishment in every household; but mental development should not be sacrificed to the acquirement of manual

skill. Since the young especially find mental effort the hardest task, they readily take to manual work which calls for little mental application rather than to studies requiring concentrated thought. It is a matter of my personal observation that mechanics and artisans who possess great skill do not always have the power of mind to apply it in the most advantageous manner. It is one thing to build a house and another to furnish it. It is one thing to have skill and another to make that skill most effective. The purpose of mathematical discipline is the development of the mind. The principles of mathematics do not even contain the interesting information that many other subjects possess. Unless a pupil is benefited by the mental training which the study of mathematical principles is capable of giving, the study of the subject must be considered a waste of time and energy.

3. That a strong mind can be developed only through the medium of clear ideas. The pupil must acquire clear ideas of every point of the ground covered if he is to derive the mental benefit which the study is capable of yielding. One clear idea thoroughly grasped by the pupil and made his own mental possession is of vastly greater benefit to his development than an imperfect understanding of many ideas. One of the most pernicious habits of the mind against which we must constantly and most vigilantly guard our students is that of superficiality, of being satisfied to take up a second idea before the first is entirely clear and assimilated. This habit of superficiality is but a prelude to a slipshod way of doing things, to being satisfied when a task has been done so that the result will look right, yet not stand careful examination. To do every task right and to the best of one's ability is essential to every occupation in life. The time to begin this is at school, whatever the subject taught. A superficial knowledge and a lack of understanding of the work is responsible in some degree for the tendency of the pupil to conceal his ignorance in his desire to appear to know what he really does not understand. Yet nothing is so detrimental to the aim and purpose of teaching; nothing makes the earnest effort to achieve this purpose so futile as a pupil's persistence in concealing his ignorance, his pretending to know and understand, instead of frankly trying to obtain all possible help in his difficulties. The

teacher should not take any knowledge of the student for granted. What appears to be self-evident to him may be entirely obscure to the student; the latter may not even be conscious that the idea is not clear to him until he is questioned about it. It is the duty of the teacher to require of the student the reasons for every step and to convince himself that each point of the work has been thoroughly understood. We can never be sure that we understand an idea until we come to express it or apply it. One means of acquiring clear ideas is to speak them out or to write them down. We must have the student talk the ideas over with us. Only in this way will he find out whether or not they are perfectly clear to him. This requires a great expenditure of time. A plan which helps to secure, on the part of the student, a thorough understanding of an idea, is the following: After carefully explaining a difficult proof or deriving a complicated formula, let the teacher erase the work on the blackboard and go over the proof or the derivation of the formula a second time in the same complete manner and with as much detail as at first. I recognize the strain on the teacher in assuming the proper attitude of mind and expending the same amount of energy in going over a proof a second time. To say the least, it becomes tiresome. This method should be followed, not only in the teaching of mathematics but of any other subject. No matter how carefully a child may read a story, or how full the explanation on the part of the teacher may be, the story will impress itself much better on his mind if it be read by him a second time immediately after the first reading.

This is not a new principle in education. Its efficacy was recognized even at the beginning of history, as is attested by the incident related in Erubin 54 b (the talmudic account of the way in which Moses conducted his school for the original teaching of the Torah and the Mishna):

"Moses repaired to his tent, followed by Aaron, to whom he communicated the received law and its interpretation. Aaron then rose and removed to Moses' right side; whereupon Aaron's sons, Eleazar and Ithamar, entered, and received the same communication from Moses, after which they took their seats respectively on either side of Moses and Aaron. Then the seventy

elders came, and Moses taught them in the same way as he had taught Aaron and his sons. Finally all the people entered, or every one who had a desire for the knowledge of the Lord, and Moses made them also acquainted with these teachings of the Lord. In this way, Aaron had heard the law from Moses' lips four times, his sons three times, the elders twice, and the people once.

"Then Moses rose from among them, and Aaron repeated aloud what he had (now) heard four times, and he left the tent. Then Aaron's sons, who had by this time heard the law four times (three times by Moses and once by their father) repeated it aloud to the audience, and they left. Thereupon, the seventy elders, who also by this time had heard the same teachings four times, repeated it to the people. The people now had heard it four times also, once from Moses, once from Aaron, once from Aaron's sons, and finally from the elders."

With the immature mind of youth, there is little danger of explaining too much, of repeating an explanation too often; the danger lies rather in the opposite course. With the young, the acquisition of clear ideas and the formation of habits of thinking clearly and logically are very slow mental processes. It is, so to speak, a matter of evolution. If instruction be carried on in such a manner, progress will be very slow, but true progress can be measured by the quality of the work only. Quantity without quality is, in education as in anything else, of little value. This should be impressed on the child and youth as the fundamental principle of their moral and intellectual development. Its neglect is a danger of national concern. The pupil must practice in his work and in his play and in all his other activities the principles of thoroughness, of honesty and of sincerity.

4. That very few graduates of high schools have the privilege of attending college. I feel that the mathematical subjects taught in the secondary schools, are, on the whole, selected and arranged with the view of satisfying the entrance requirements of the colleges and engineering schools. On this point, I wish to state that while it is my fervent wish that each and every youth, if he is by inclination and ability fitted for it, should receive the benefits which ought to result from a college

education. I believe that the secondary school has a distinct place in an educational system. We cannot arrange the curriculum of the secondary school with a view to the needs of the few. We cannot teach mathematics in such a manner as to enable the one girl out of ten millions to use it if she will become a surveyor or an engineer. All our teaching must be done with the single view of what is best for the majority and of what will inspire all the pupils and help all of them to true happiness and high ideals.

I do not know by what process we have arrived at the present curriculum in mathematics in the secondary schools, at least as far as the amount of subject matter ordinarily given in these schools is concerned. Even if the amount which a class of average ability is able to cover in a given time is taken as the normal, it must be recognized at once how difficult it is to generalize from such assumptions. The qualities of the teacher, the environment of the pupils, the attention they give to the instruction, the time they spend in preparing the lesson outside of the class room, all these must be taken into consideration, for they make the amount which a class can be expected to cover in a given period of time, a variable one. On that account, I feel that a prescribed, fixed course to be followed by all students in a subject, is an impossibility in education. If the teacher is competent and able, then he and he alone should determine how much of the subject the pupils of his class are able to assimilate without overtaxing their strength. I realize that this presupposes a teacher who does his best without supervision or direction. Even in the same institution, in the same subject, the mental average of one section may differ from that of another. In teaching the same subject to two different sections, I have found myself able to cover considerably more ground with one section than with the other. As I have said, I have been wondering how the subjects comprising the curriculum of the high school have been chosen. It is not to be supposed that one man with equal mastery and knowledge of all subjects of the curriculum and without prejudice in favor of any one of them, allotted the time to be given to each in proportion to its difficulty and the part it plays in developing the different qualities of mind. I feel, therefore, that the curriculum

is a compromise between the different branches of learning, and hence is necessarily, to say the least, a make-shift.

A distinct indication that students of the secondary schools cannot, as a rule, acquire clear ideas of the mathematics ordinarily studied in these schools is the generally acknowledged fact that most of the graduates of the secondary schools do not possess clear ideas on these subjects. I think that most of us will readily assent that many of the ideas we have learned at school were not entirely clear to us at that time. The reason in most cases was that we were not allowed sufficient time for the ideas to grow on us, so to speak, or we were not mature enough to gain perfectly clear ideas of the concepts presented to us. It takes, for instance, greater maturity than is possessed by the average high school pupil and more time than is allotted to him, thoroughly to understand some of the principles of demonstrative geometry, and to acquire facility in solving constructions. It is also questionable whether under the present system of instruction it is possible even for students with normal ability to acquire a clear conception of each of the ideas taught at school. To accomplish it, the pupils of a class must all be of the same age and of the same ability. But if pupils of different ages and of different mental powers are grouped together in the same class, are taught by the same method, and are expected to cover the same ground, the possibility of all of them acquiring perfectly clear ideas of every subject taught is very small.

The student must thoroughly understand the meaning, the philosophy, so to speak, of each mathematical concept presented to him. It takes him a long time to familiarize himself with the concepts of any mathematical subject and to acquire ability and facility in applying its principles readily. Few students have a clear understanding of the quantitative significance of the theorems of proportion, such as. "A line drawn parallel to a side of a triangle divides the other sides proportionally," or, "Similar triangles are to each other as the squares of the homologous sides." In all my experience, I have not received from a pupil a satisfactory explanation of the truth that one divided by infinity is equal to zero, or any number divided by zero gives infinity, both of which are conceptions used in

secondary mathematics. The young find thinking a hard task. A pupil will undergo the labor of memorizing a proposition and its proof rather than follow the more difficult course of concentrating his mind and trying to understand the meaning of a principle.

As I have said before, the teacher should not present a new idea before the pupil has an intelligent and thorough understanding of the idea under consideration, and has acquired sufficient facility in handling and applying it. Such a course requires careful instruction, and long meditation and reflection. Not to be misunderstood on this point, I wish to say also while any branch of mathematics consists of a sequence of principles which are joined together like the links of a chain, it is not always possible to gain at once a clear understanding of an idea or of the proof of a principle, yet this principle may be needed in the further development of the subject.

Expediency may make it advisable to postpone a proof of a principle until a later date, but under no circumstances should an easy but fallacious "proof" be substituted for the rigorous one. Even if one can give to a student a fairly clear idea of the logical basis of a particular algorithm by using much time and effort, it is often the part of wisdom to defer such work until the student is more mature.

In plane geometry, the teacher will have to exercise great care in presenting this subject if he would not discourage and confuse the beginner. It is advisable to omit the question of the incompleteness of the axioms and the correctness of the definitions, and conceptions of the non-Euclidean geometry. Euclid's proof of the theorem that the sum of three angles of a triangle is equal to two right angles depends on the fact that through one point outside of a straight line, only one parallel is possible. By intuition, the pupil is convinced of the latter fact and to make him see that the opposite is conceivable might tend rather to confuse than to clarify the ideas in the immature mind. On the other hand, the average student will not be satisfied with the mere statement that to subtract a negative number is equal to adding a positive number of equal absolute value.

In viewing the subject of the secondary mathematics from

a higher level, there are no distinctive lines of demarcation between the different subjects, between algebra and arithmetic or between analysis and geometry. But to the beginner it does not appear so. For him each subject has a new language, a new symbolism, new notations and new ideas. When after studying algebra, let us say, the pupil begins the study of geometry, he meets with ideas all differing from those he has acquired before. Even the ideas of solid geometry, although related to those of plane geometry, need for their study faculties of imagination different from those applied to the latter. True enough, trigonometry does not contain any new fundamental concepts which are different from those which the student has learned in algebra and geometry but only a new and more convenient notation which enables us to obtain new relations between the old concepts. Yet it takes a long time for the beginner to familiarize himself with the new nomenclature, with the new notation, and with the mode of representation of the ideas of the subject.

In making up the curriculum of mathematics this fact ought to be taken into careful consideration. Schools which teach all of the four branches are teaching, as far as the pupil is concerned, four different subjects, as different as four different languages.

In order to acquire a thorough mastery of any branch of mathematics, the student must be able to perform with facility and little effort such operations from other branches as may be needed in this. When he takes up the study of algebra, he must be able to perform with ease such arithmetical operations as he may need in algebraic work. In the study of trigonometry, he must have perfect facility in performing such algebraical operations as may be required there, just as, when we have acquired facility in spelling and writing, we need pay little attention to these, but can concentrate our efforts on the formulation of our ideas. To carry a mathematical operation through all of its stages to a correct result requires a thorough knoweldge of the subject and a facility in performing all its operations. This can be acquired only by constant practice, which again takes a great deal of time and effort.

It is true that the different mathematical subjects have each

the ability to develop peculiar qualities of mind. On the whole, however, this varied ability is greatly exaggerated. The student should confine his energies and his time to the study of fewer subjects and acquire a thorough knowledge of their ideas, rather than divide his energies and abilities among a number of subjects and run the risk of gaining only a superficial knowledge of them. I see no reason why the study of numbers cannot be used as a principal tool, as far as instruction in mathematics is concerned, for developing power of mind, especially with those who will not continue their studies beyond the high school. Numbers are certainly more real to us than any of the other concepts of mathematics. We are most familiar with their use. Their study is also very fascinating and is peculiarly fitted to develop reasoning power. We sometimes unconsciously attribute to a person who has studied geometry and trigonometry a broader and better education than to one whose knowledge only extends to operations with numbers. But a knowledge of some of the properties of the triangle and of the relations of the trigonometric functions has no more cultural value than that of the properties of decimal fractions and of periodic decimals, of the different tests of divisibility of number, etc. While the study of any mathematical subject can be made to serve from its very beginning the purpose of developing the mental faculties in the learner, yet these are best strengthened and brought into play when the pupil, with a clear understanding of the fundamental principles, proceeds to the further development of the subject. The amount of mental development is therefore commensurate with the student's familiarity with the subject, that is, the more intensive the study, the greater is the mental development. There is also a great deal of mental training to be derived from solving, by arithmetical processes, some of the problems ordinarily given in algebra. This is an invaluable exercise for developing power of mind; and it helps the student to appreciate the short methods available by the use of algebraic notation. It is not the number of subjects which a pupil has studied at school but the mental benefit which he has derived from the study, that counts.

Some work is done in the school because it develops the

qualities of neatness, of being orderly. But these qualities are much more common than the quality which is the principal purpose of the school to develop, namely, power of mind. It is comparatively easy to secure the services of people who are able to do routine work with neatness and accuracy. But it is difficult to find a person to fill a position in which mental power is necessary and indispensable.

Some mathematical subjects are taught in the high school with little regard to the mental training which they are capable of yielding. Take, for instance, the subject of trigonometry. The course in trigonometry as given in the high school consists mostly in the solution of triangles which involves work with logarithms. This the student performs in a mechanical manner induced by the use of tables.

From my experience in teaching classes, extending over more than two decades, and from my experience in the instruction of private pupils, both delinquents and others, I am satisfied that it is not possible for the average pupil to cover the course of study in each of the mathematical subjects as required by the secondary schools and as given in the text-books in use, and at the same time to gain a thorough knowledge of each and every one of the ideas presented. I consider it entirely impossible that algebra, for instance, can be assimilated by the average pupil at the rate of one page a day, especially since most of the text consists of examples and problems. I consider it equally impossible to study geometry at the rate of two pages a day if any exercises in geometrical constructions are included. The purpose of studying any subject is, as I have said, the acquisition of mental power and initiative. Take, for instance, the subject of factoring. The good which should come to the student from such work is the development of such power, facility, judgment, and initiative as will enable him to work examples differing from those which were worked for him. It is the power of initiative which he must obtain. The same is true of the solution of problems. It takes a long time for the student to acquire such powers as are necessary to express the conditions of a problem in the signs and symbols of algebra and effect its solution.

In this connection, I wish to state that an examination which

consists of work the student has done or gone over before is, on the whole, not the proper test of the mental power which he has acquired by means of this work. It is the duty of the teacher to see that the pupil, as the result of his training, is able to apply the power obtained to work which he has not seen or done before. Such work must, of course, be only within the limits of his acquired mental strength. It is with the utmost reluctance that I also state that an examination is only a true test of the knowledge of the student when it is searching, that is, when the student is able to depend on his own knowledge and ability. A pupil who is wholly honorable in all his dealings outside of the classroom and aside from his relations to his teacher as a pupil, will not consider himself guilty of wrong-doing in deceiving his instructor. Also, under the present system of marking, as I understand it, it is possible for the pupil to get a passing mark in any of the branches of mathematics by having hazy ideas on many points, without having a perfect knowledge or a clear idea of any one of the parts or topics of the subject. Most examination papers in geometry, for instance, are of such a nature that the student may pass them if he is able to prove theorems given in the text-books and without his having acquired facility in solving constructions. An examination, if it is to be searching, and no other should count, should be oral.

The school should set for its solution only such problems as it can solve. Such a course will have not only a beneficial effect on the pupil's mental development, but also on the moral. It is alarming to think that although the average high school pupil is not able to acquire a thorough knowledge of all the subjects of the curriculum, there is a tendency on the part of the high school to extend the latter so as to include some of the subjects usually taught in the college. There is more and more overlapping of high school work and college work. If the former would confine itself to as much work as it can do most efficiently, the true purpose of education would be better attained. In this connection I wish to state that the college is just as much to be condemned if its undergraduate curriculum includes more than the average student is able to assimilate, or if it includes courses which ought to be offered in the graduate school only.

I believe that one of the most important functions of the school is to teach the pupil to acquire habits of diligence, of finding enjoyment in work. A greater time must be given, especially to the young, for recuperation from mental work than from manual work. The amount of time which can be advantageously applied in concentrated mental effort is much smaller than the time during which a person can do manual work. A lack of appreciation of this fact works a great injustice to pupils, to teachers, and to investigators. The secondary school pupil, at the age when he attends school, should not be expected to do mental work for a longer period than six to seven hours a day, and this should be broken into short periods with rest periods intervening. We must not crowd into the mind of a pupil too many ideas during one and the same period of instruction. After an idea has been explained to him, he should be given an opportunity to recover his mental composure, so to speak, before taking up another idea.

The tendency, very properly, is to reduce the number of hours during which a man shall work at manual labor. In fact, in some countries, laws are enacted which limit the number of working hours to seven or eight per day. And these enactments are for the benefit of able-bodied men who do manual labor. How much more important, then, is it to regulate wisely the number of hours during which the boy or girl may be employed, especially in mental work. The average time during which the pupil in the high school is expected to be engaged mentally, is a little over five hours per day. To this must be added the time which the pupil is supposed to spend outside of the school with homework. However, interesting the instruction, however cheerful and well-ventilated the class room, the strain on the pupil's mind, the strain on his senses and the strain on his body from sitting quietly four or five hours make imperative a longer recuperation of mind and body, if the health and the physical development of the youth is not to be impaired.

The amount of ground covered in a subject does not determine the mental status of the learner. The mental powers of a person cannot be measured by the number of books read, by the number of subjects studied, by the number of facts

learned, but only by the number of clear ideas acquired, and by the extent to which he is able to apply them to his own moral and physical betterment, and to that of his fellowmen. If the pupil thoroughly understands every idea presented, and acquires a perfect mastery of each process employed; if he is able to solve with facility and accuracy every problem taken up; then, whatever may be the amount covered daily, however small it may be, he will grow in mental power from day to day. With such a method of instruction, progress must necessarily be very slow, but the knowledge acquired will be thorough, and the steady development will give strength and depth of mind. Under such a system, even the dullest pupil will acquire some power and some knowledge of the subject.

In formulating a course of study in any subject the principal question should be, how much ground can be covered by the average student in so thorough a manner that he may gain a perfectly clear idea of each and every point included. I know how difficult it is to break with established traditions. It will take the united efforts of all of us to bring about a change from existing conditions, to reduce the amount of each of the mathematical branches taught in the secondary schools and in colleges, and to include in the curriculum only as much of each subject as the student is able to make his mental own.

There is an idea which I fear is not sufficiently appreciated. The student, is as a rule, left under the impression that the mathematics which he is studying in the ordinary course, as given in the best technical schools, will prepare him to solve all the problems which he might meet in the practice of his profession. It is true that most engineering problems need for their solution only simple mathematical devices. Some of them do not require even a knowledge of the principles of calculus. On the other hand, there are many engineering problems which require a wider knowledge of mathematics than the student covers in his course. One of the most powerful instruments for solving engineering problems and those of other applied sciences is the differential equation. Owing to the lack of time and the lack of development of the average technical student, a course in differential equations is not included in the curriculum of most of our technical schools. The solution of the

different equations which arise in connection with some practical problems is often quite difficult. Their solution taxes the ability even of the ablest mathematicians. To solve such problems, the student will have to study this and similar mathematical branches. But he will be much better able to take up the further study of mathematics if he has perfectly clear ideas and facility in the mathematics which he has covered, even if their scope is limited. If he lacks this perfect knowledge and facility, he is at a great disadvantage.

My sympathies are with the competent teacher of mathematics in the secondary schools. Even those of them who have the best qualifications for their calling must recognize the hopelessness of their task, to meet the requirements of the college and at the same time have their pupils acquire thoroughly clear ideas on each of the subjects gone over. With many of them, the tenure of office depends on the number of men who are drilled but not prepared to go through the examination papers as set by the college and acquire a sufficient mark to pass. The whole energy of the teacher, I am sure much against his better judgment, is directed toward the one aim,—that the student pass the entrance examinations to the college.

We, the teachers in the college, are to be blamed for such a state of affairs; it is in our hands, by reducing the amount of mathematics required for admission to the colleges, to make it possible for the earnest and capable teacher to do his work to his own satisfaction and to the best interest of the pupils. The schools will be only too glad to comply with our wish, as it will relieve them of a burden which they cannot carry and it will make it possible for them to do their work in an effective manner.

I understand that in some schools special classes are formed for those who wish to enter college, but the majority of schools, especially the public high schools, have not sufficient funds to separate those who may wish to enter a university from those who do not; and I know of at least one case where a whole class, in a small city, was made to go through all the work required for admission to one of the leading institutions in this land, because one boy wished to enter this institution.

The curriculum of the secondary school which helps the pupil

to acquire maximum mental power should be that which best prepares him for college entrance or for any other work. The colleges will have to modify the entrance requirements in such a manner that more emphasis is placed on mental power acquired and less on detailed knowledge of a subject.

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DISCUSSION ON PROFESSOR SCHWATT'S PAPER.

Mr. E. D. Fitch said:

I may say that I am in hearty sympathy with the ideas presented in the paper. The problem in private secondary schools, however, is modified by the fact that such schools are very largely fitting schools for college, in that the college is the goal of a majority of their students. The element of time, determined by the desires of parents and by the standards of college entrance examinations, makes it impossible for the average student to thoroughly understand all the mathematics taught, or to follow to the letter the suggestions of the paper, valuable as such suggestions may be.

We are expected to prepare a boy to pass the college requirements for admission without proper consideration, in many cases of the maturity of the student's mind. It is easy for the college to criticize the lack of preparation of the boys entering college; it is equally easy for the secondary schools to pass the criticism back to the elementary schools. The college constitutes itself a court of last resort as to the efficiency of mathematics teaching in the secondary schools; but may it not be fortunate that the college has no court of last resort as to the efficiency of *its* instruction? My knowledge of some of the work done by college students leads me to feel that it does as little credit to their instruction as that done by secondary school students.

Mr. E. B. Ziegler said:.

Doubtless every teacher in this room today looks at the problem under discussion from the broad standpoint of life. That is, we feel it our province to prepare our students as fully as possible, for social and industrial efficiency; and we believe

that every activity of the class room should be directed toward this end. The several speakers this morning have this general aim before them. But the particular aim of each is determined by the institution in which he teaches; so that the problem presents a different phase to each one of us. Two of the preceding speakers have looked at it from the college man's point of view; one from the standpoint of the private preparatory school; but I must take the viewpoint of the public high school. The college must solve the problem of preparing the student for some special pursuit or profession; the private school has the fairly definite task of preparing pupils for college entrance; but the problem confronting the public high school is decidedly complex. The public high school is, first and foremost, the people's finishing school. It must give a broad and general training for life. A glance at the courses of study in representative high schools shows that these courses have been changed to meet this demand for broader training. We still expect the high school to be a college preparatory school, but since less than five per cent. of its students enter college, its curriculum must be diversified; and a high school of any standing at all offers three or four courses, showing a demand on the part of the public for other work than preparation for college.

But all these added and diversified courses mean more work for the public high school, and perhaps less time for any one particular branch, as mathematics. Dr. Schwatt has shown that in his opinion, the boys are not given the proper sort of mathematical training in the secondary schools. We have heard this statement from college men for a number of years, and they usually base their assumption on the work submitted by the boys in their entrance examinations, and upon their later work in college. But are the colleges justified in drawing from these data the general conclusion that the mathematical training in public high schools is weak? In the first place, they receive less than five per cent. of the public high school pupils; moreover, there is usually a gap of two years between the boy's definite and intensive work in algebra, and his college entrance examination in that subject, and a gap of at least one year between his work in plane geometry and the examination.

Under these conditions, it seems hardly fair to say that the college entrance examinations prove that the secondary schools are, or are not, securing satisfactory results in mathematics. Nor is it quite fair to say that the work done in college mathematics is a fair test of secondary school work. College algebra is not given until the second half of freshman, or the first half of sophomore year, and there has been an interval of two and a half or three years in which the boy has not done any definite work in this subject except during the few weeks in which he crammed for his entrance examinations. But, while I felt it unfair to test the efficiency of the secondary schools by work submitted a long time after the completion of that special subject in the secondary schools, still I was by no means sure that mathematical training in our high schools is as thorough as it ought to be.

In order to satisfy myself, and not wishing to speak out of my own narrow experience in a borough high school, where I touch only about one hundred students of mathematics during the entire year, I prepared a questionnaire, and submitted it to about thirty representative high schools in Pennsylvania, including some of the smaller as well as the larger schools. In this questionnaire I asked the following: Are algebra, plane geometry, solid geometry or trigonometry electives in your course of study? What per cent. of pupils fail in each? What proportion of pupils receive an average of less than eighty per cent.? I asked the last question because I base my opinion of thoroughness upon a percentage of eighty. Surely a pupil who falls below this mark in any subject cannot be called thorough in that subject.

The replies show that 18 per cent. of the pupils fail in algebra, 17 per cent. in plane geometry, 8 per cent. in solid geometry, and 5 per cent. in trigonometry; while the test for thoroughness shows that 47 per cent. lack thoroughness in algebra, 43 per cent. in plane geometry, 37 per cent. in solid geometry, and 38 per cent. in trigonometry. Surely these percentages show that Dr. Schwatt is justified in his conclusion that the secondary schools are not giving a thorough training in mathematics.

Having stated this fundamental proposition let us analyze the situation, and try to discover the factors that tend to establish

the validity of our proposition, and then let us attach the responsibilities as we see them, to each factor. There are a number of factors which we ordinarily hold responsible for the weakness in secondary mathematics, and we must inquire how far we are justified in fastening responsibility to each factor.

First, to what extent is the teacher in the elementary school responsible? We secondary teachers are prone to shift responsibility to the elementary school, and to assert that the pupils come to the high school with very vague mathematical ideas; that they are not thoroughly grounded in principles in arithmetic, and that it is impossible for us to build up the high school mathematics on such a weak foundation. But have we ever considered whether or not we are justified in this old-fashioned notion that algebra and geometry are built upon the basis of arithmetic, or that any one of these three branches of mathematics requires the same mode of thinking as the others? Personally, I believe that each of these subjects requires a different mode of thinking. I believe, too, that the mechanical operations of each are very different, and that the subject matter of any one has no relation, except an artificial one, to the subject matter of the other two. Arithmetic is concrete, or can usually be made so. Algebra must remain largely symbolic. The constructions in geometry are concrete, to be sure, but geometry is related to only one phase of arithmetic—mensuration. How does the working out of problems in percentage, in common or decimal fractions, in denominate numbers, contribute to the work in algebra or in geometry? Even the four fundamental operations when carried over into algebra loose much of the arithmetical significance, for the element of sign is added and the processes of borrowing and carrying are eliminated. The child who is able to do a hundred problems in arithmetical subtraction without a single error, may be the very last one in the whole class to master subtraction in algebra. On the whole then, I believe we are safe in saying that the different branches of high school mathematics have little in common with elementary arithmetic, but that each branch requires the formation of a specific habit of thought peculiar to that branch.

So then, we must in all fairness, take a little of the responsibility from the shoulders of the elementary teacher. There is,

however, one phase of mathematical training which we must demand of her. We must insist that in the elementary schools the pupils acquire the habit of mechanical accuracy, and the spirit of industry. For, while we may not say that the high school boy fails to understand algebra because he has never been rightly trained in arithemetical principles and reasoning, we have a right to say that very often he fails because he has not been taught to apply habits of industry and accuracy to mathematical work in the lower grades.

The pupil, then, leaves the elementary school with a mode of thinking peculiar to arithmetic, accurate or otherwise, with a spirit of industry or without it. He comes to the high school, one of a group of fifty or more, at the age of adolescence—at a period in life when his whole body is undergoing a tremendous physical change—when new interests are continually claiming his attention. This is the period when the average boy looms big in his own vision. He has left the grammar grades, and is now one of the big fellows. It is up to the mathematics teacher of the secondary school to take a large group of these boys, each with a greater diversity of interests than he ever possessed before, with possibilities probably greater than ever before, and to help each boy in the group to form new habits of mathematical thinking. It is at this particular period that the boy needs the best teachers in the whole corps. The teacher of mathematics should know not only his subject, but should understand the psychology of adolescence. Yet it is just at this period, the first two years of high school, that we often find the greatest weakness in the teaching force.

I believe that if our pupils are not receiving a thorough training in secondary mathematics, that the great fault lies in the secondary schools themselves. The two great factors in the student's development are the content of the curriculum and the teacher; and since in many cases, the teacher is at liberty to select his own content, the fundamental responsibility must rest with him. What, then, are some of the weaknesses that keep him from measuring up to the situation? I believe that the scientific attitude is often carried too far in our high schools. A large percentage of our high school teachers call themselves specialists and approach their classes with a scientific attitude

which insists upon the pupil noticing every little detail of the subject under discussion. The pupils go from the class room with a lot of confused and vague ideas instead of some one definite principle thoroughly mastered. Too many teachers, particularly those fresh from college, follow the lecture system in imitation of college class room method. But this system is not adapted to secondary teaching. In order for the boy to get a maximal development, he must be encouraged to dig for himself. The lecture system lets him off too easily. Still, we must not be too hard on the teacher in this matter. Usually his classes are entirely too large for effective work. Probably he feels that with classes ranging from thirty to fifty, the lecture system is the most economic.

Another point of weakness which we have hinted at, is the arrangement of our high school teachers. Too often the new and inexperienced teachers are given the large classes of freshmen and sophomores, while the experienced teachers are placed over the smaller classes of juniors and seniors. But isn't it a great mistake that the teacher with less experience, and with less understanding of boy life should be placed over adolescent boys, at a period of their greatest possibilities, when they are most susceptible to the formation of new habits, and when a new mode of mathematical thinking is introduced? The result too often is a poor start in high school mathematics, and a consequent dislike for the entire subject on the part of the boy.

I have granted that there *is* a lack of thoroughness in secondary mathematics, and I have tried to point out some of the reasons for this lack; now may I suggest some remedies? First, our classes should be smaller. Every student should feel that he is apt to be called upon to recite every day in the week. The boy's feeling of personal responsibility is often lost in large classes. Second, there should be elimination of certain subject matter—the non-essentials. Let us give up the little side excursions into caves and caverns where the light is too dim for the boys to read the inscriptions on the walls, and let us keep to the broad highway. I believe we should take more time to emphasize the fundamental principles. I am not alone in this opinion. Three fourths of the letters received in answer to my questionnaire suggest that a certain part of the work usually given in

each branch ought to be eliminated, and that more time should be spent on essentials. Third, the teachers should have that sort of pedagogical training which gives insight into the best methods. They should be in sympathy with the student, and in touch with life; and only experienced teachers should be given to the lower classes. Fourth, there should be close supervision of the secondary schools, so that the various subjects shall have the proper amount of time, and that the course of study may be well balanced. If these four conditions are fulfilled, I believe that the question of thoroughness in secondary mathematics will be solved.

Mr. H. A. Foering said:

In connection with the assertion that "The average secondary school pupil is not able to acquire a thorough knowledge of all the mathematics ordinarily given in these schools," I wish to say that I do not believe it is right to hold secondary school teachers responsible for all the failures in mathematics in college. Unquestionably, there are many instances where students have been thoroughly taught in these fundamental branches, have digested the principles, and after the lapse of a year or two, when their college work demands a knowledge on their part of some of these principles, the student shows himself very deficient in these very things. I wish to call attention to a conversation I had with the head of the civil engineering department in one of our best technical colleges. He said to me, "Mr. Foering, boys that come to us from secondary schools are not able to use logarithms accurately and with facility and to solve triangles. I do not believe it is possible for secondary schools to teach this matter properly." I answered him, saying, "Professor, you are wrong. I extend you a hearty invitation to visit our school, and I believe you may do so in most other schools, at the end of our course in trigonometry. You may come in, take a class for the whole period, subject the members to an oral or written examination, and test them on their knowledge of the subject and their facility in the use of the tables and the solution of the triangles. I feel sure that with a few exceptions you will be astonished at their skill. Some of these boys will enter your college and take your civil engineering course. Six

months or a year later, when you want these boys to make use of these tables, before proceeding, kindly put them through the same test you put them through in our school. You will find that a great deal has escaped them, that they have lost their facility through lack of practice and forgetfulness."

This same thing is true of the fundamentals on which rest certain subjects in analytical geometry and calculus. The larger proportion of the boys have undoubtedly known these principles and forgotten them. I do not think the secondary school teacher should be held responsible for what the student fails to retain until he needs it in college. Take the subject of plane and solid geometry. There has been much said about their cultural value, their value in the development of thought power, the training afforded in logical reasoning. If these things are so much more important than the subject matter, then why hold the student to account for his lack of knowledge of all the material. In my opinion, and I have told my classes so many a time, it does not matter if on the day they graduate from school they forget most of the geometry they have learned, provided they have a sufficient mental equipment to undertake college work and have improved their logical faculties. When, later on in college, they need some forgotten reference, they may then take their text books and refresh themselves, in case they cannot reason it out without such reference. The suggestion seems good that college courses in mathematics have reviews, immediately prior to the time that it is necessary for the student to be fresh, in such elementary work as is of more than ordinary difficulty entering into particular subjects. Thus, a review, to be done perhaps in a single recitation, of such a subject as "Convergency and Divergency of Series" just preceding the study of subjects requiring their use, or of "Undetermined Coefficients," or "Decomposition of Fractions," or "Logarithms" and similar subjects might obviate much of the criticism that such subjects have been poorly taught originally. It is gratifying to observe that some of our later texts in analytical geometry and calculus are inserting such reviews.

Mr. Charles F. Wheelock said:

This is a very sad occasion. The atmosphere is charged with gloom. I have frequently attended funerals, but I do not

remember ever to have been present at a more mournful occasion than this. The writer of the paper and the six gentlemen who have discussed it before me all agree that we are failures as teachers of mathematics. They all agree that the students in our high schools are not getting anything worth while in their study of the mathematical subjects in the high school. They differ somewhat as to their interpretations of the reasons for the condition, but they agree substantially as to the condition. If we are to believe them, we have most certainly and most dismally failed. The failure may be due to the unreasonable requirements of the colleges, to the improper text-books used, to the grandfathers and grandmothers of the children, to the distractions of society, or to some other undiscovered cause, but the speakers, so far, agree that the result of our teaching is failure.

I am wondering if we feel like dropping this discussion at this point in this atmosphere of gloom, to go to our homes believing that there has been no result worth while for all the efforts that we have put forth in our attempts to teach mathematics. I for one feel like protesting against the conclusion reached. The very question which constitutes the topic of Dr. Schwatt's paper seems to me to be self-evidently absurd: "Is the average secondary pupil able to acquire a *thorough* knowledge of all the mathematics ordinarily given in these schools?" The answer must be in the negative and there is no reason for us to feel discouraged because the answer is in the negative. I doubt if we who have pursued not only the secondary course in mathematics, but who have subsequently taken a full college course in mathematics, have acquired a *thorough* knowledge of even the high school mathematics. I doubt if there is a single teacher of geometry here present who has not within the past year discovered some new bearing to a proposition in geometry that he has been teaching for years and that he thought he formerly understood, and this growing of his knowledge will continue as long as he continues to teach these subjects. The student in algebra does not have a thorough knowledge of the full, broad meaning of an equation until he has studied geometry and, I might add, until he has studied analytical geometry and calculus. Each step forward in the

course puts him in a position to see the previous steps in new relations and in larger perspective. That the immature mind of the high school student has not thoroughly grasped all the bearings of his elementary mathematics is not a condition that should lead us to discouragement and that should necessarily lead us to change our methods, for no method can accomplish the impossible. I believe that we are getting, on the average, as good results as we ought to expect to get from the teaching of high school mathematics. I do not believe that we shall ever reach the time by any method or by any amount of effort when the ordinary student at graduation from the high school will be able to do very much original work in geometry. He is not ready for that sort of thing. If the colleges require an examination in originals in geometry, they are bound to fail to get it, because the thing is impossible. We have heard from the president of Villanova that 75 per cent. of those coming up for admission to that institution failed in their entrance examinations in mathematics. I have been wondering what becomes of that 75 per cent. Do they enter the college or do they not? Does he refuse them admission because of their failure in mathematics, or do they go on just the same with perhaps a temporary halt to make up a condition?

Very many years ago, when I was a student pursuing the study of the calculus, I came across an anecdote concerning Leibnitz and one of his students. Leibnitz had just developed the new method and was instructing a young man in its mysteries. The young man had acquired considerable facility in performing the operations in a mechanical way and in getting results that seemed to be correct, but he did not fully understand. There was a region of fog somewhere between the premises and the conclusion in every example. He went to Leibnitz, explained to him his difficulties, and proposed to drop the subject because he feared he should never master it since he did not fully understand it. Leibnitz is reported to have said to him: "Never mind that; keep right on with work; just have faith and after a time the mists will roll away." This anecdote impressed me greatly because at that time I was exactly in the condition of the student in the story. I had reached a point where I could perform the operations of differential

calculus. I could develop functions into series, I could solve problems in maxima and minima. I could do many of the things that students in calculus are called upon to do, but between the beginning of each operation and the end, there was a section that was foggy and I had felt like giving up in despair and the advice of the master came to me at an opportune time. I think that I profited by it. Is not the condition stated in this anecdote a very common condition? Is it not almost universal? Do not nearly all the students of mathematics have a period in which the reasoning is not fully apprehended? Does it not require time and repetition and faith for the thorough mastery of even some of the more elementary principles of mathematics, and ought we to be in any way depressed because the immature high school student has not acquired that more complete mastery that we have after long years of contemplation?

Perhaps the college professor is the least qualified person in the world to pass judgment on what a high school pupil ought to know. He has been accustomed to deal with more mature minds and he is apt to be disappointed because the student at his entrance to college has not the full conception of mathematical principles and the power to grasp new ideas that he is expected to have at the end of his college course. To my mind, some of the college courses in mathematics are very unreasonable. In one large university with which I am acquainted, students in engineering are expected in the freshman year to cover analytical geometry, descriptive geometry, differential and integral calculus. In a recent conversation with several graduates from that course, it was distinctly stated that they knew nothing whatever of the calculus and that they never knew anything about it except just enough to pass the examinations. If the high school could have the privilege of examining the college graduate, probably as good reasons could be found for asserting that colleges are not building well upon the foundation laid in the high school as are now found for the assertion that the colleges can not do their work well because the foundation has not been well laid in the high school.

My friends, whatever we conclude regarding these matters, do not let us go home thinking that we are absolute failures. Let us look over the field and try to find some points wherein

we have succeeded. I believe that we shall find them if we look for them. We would be in a better frame of mind for carrying on our work if we are looking for the successes rather than looking for the failures, and believing in success rather than in failure.

Mr. Eugene R. Smith said:

It seems to me that there can be but little doubt as to the main contention of Dr. Schwatt's paper, namely, that the average pupil does not become thoroughly conversant with all his secondary mathematics. I am, however, surprised that none of the speakers have touched on what I believe to be one of the chief causes of this condition—the distractions of modern life. The successful study of mathematics presupposes at least a fair degree of concentration, and the home and social conditions of many of our pupils are such that the chance of their acquiring the ability to concentrate on school work is very small. What has been said about mathematics is not inherent in the subject, it can with equal or greater truth be said about any study in the secondary curriculum.

One of the causes contributing to this danger from distractions is the weakness of too many of the American parents, which results in a deplorable lack of co-operation with the school. I will quote you a typical case; a father called on a school principal to ask his permission for his son to take a trip that would necessitate his losing some days of school. The principal, finding the boy's record poor, refused his permission, whereupon, to his astonishment, the father said "I am very glad that you refused, for I do not think my son ought to go, *but I had to promise to ask you.*" What the parent is unable to do, the omnipotent teacher has forced upon him.

There are a few details in the various papers to which I would like to call attention.

Something has been said about a school that was handicapped by the use of a syllabus. Such trouble could come only from the teacher's lack of understanding of the purpose and proper use of a syllabus. With the great number of colleges for which schools are preparing students, and the varying entrance requirements—sometimes even determined by the whim of one man

—an absolute definition of the subject matter—in other words, a syllabus—became necessary. Such syllabi have been, or are being, prepared in the different subjects of mathematics. Properly used they will serve as a guide to what should be taught, and there is no reason why they should weaken or circumscribe the work of any teacher.

One of the speakers has urged more mental arithmetic; let me add, more mental algebra, and more mental geometry. In my opinion, pupils can hardly do too much work in which, if I may use that expression, the teacher actually sees the pupil do his thinking. The ability to work out even a simple question from the mental picture, without the aid of pencil and paper, is a valuable asset in itself, as well as splendid training for other faculties.

The percentages of which we have heard seem to me to mean little or nothing. They may be very good or they may be very bad. It all depends on the severity of the course, the examination, or the teacher doing the marking. Unless the measure used it the same, comparative measurements mean nothing.

To come back to the subject under discussion, if this condition exists, what can be done about it? That is a hard question. Much has been done, and is being done, to lessen the amount of work required. The National Geometry Syllabus Committee will soon publish a report that reduces very materially the number of theorems in plane and solid geometry. It also recommends that plane geometry be given one and one half years, during which time the elements of solid geometry shall also be taught. This would help the matter as far as geometry is concerned, but can we get the year and a half? Probably not, for, as you all know, the other departments are also fighting for more time.

As far as I can see, therefore, the only solution is to omit the least important parts of each subject, as the geometry committee is doing, and to teach the rest to the best of our ability, believing that the pupils are obtaining a great deal of valuable training, even if, as their immaturity makes likely, they never become perfectly familiar with all we would like them to know.